Length, Green's Theorem, and Path Independence

Exactness and the Existence of a Primitive

Lecture Notes

November 3, 2026

1 Length and Integration with Respect to Arc Length

Definitions

$$\ell(\gamma) := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_a^b |z'(t)|\, dt,$$

$$\int_\gamma f\, ds := \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt, \qquad \int_\gamma f(z)\, |dz| := \int_a^b f(z(t))\, |z'(t)|\, dt.$$

Both are **independent of parametrization and orientation**.

Lemma

$$\left| \oint_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, |dz| \ \Big(\le \sup_{z\in\gamma} |f| \cdot \ell(\gamma) \Big).$$

Proof.

$$\left| \oint_\gamma f(z)\, dz \right| = \left| \int_a^b f(z(t)) z'(t)\, dt \right| \le \int_a^b |f(z(t))||z'(t)|\, dt$$
$$= \int_\gamma |f(z)|\, |dz| \le \sup_{z\in\gamma} |f(z)| \cdot \ell(\gamma).$$

□

Let γ be an arc.

Lemma (uniform convergence)

Let $f_n, f : \gamma \to \mathbb{C}$ be continuous, $f_n \rightrightarrows f$ uniformly on γ. Then $\oint_\gamma f_n(z)\, dz \to \oint_\gamma f(z)\, dz$.

Proof. $\left| \oint_\gamma f_n(z)\, dz - \oint_\gamma f(z)\, dz \right| = \left| \oint_\gamma (f_n(z) - f(z))\, dz \right| \le \sup_\gamma |f_n - f| \cdot \ell(\gamma) \to 0.$ □

Corollary

Let $f_n, f : \gamma \to \mathbb{C}$ be continuous with $\sum_{n=1}^\infty f_n = f$ uniformly on γ. Then

$$\oint_\gamma f(z)\, dz = \sum_{n=1}^\infty \oint_\gamma f_n(z)\, dz.$$

2 Terminology

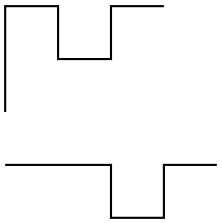
γ is:

(1) **simple**, if $\gamma : [a, b] \to \mathbb{C}$ is injective;

(2) **closed**, if $\gamma(a) = \gamma(b)$;

(3) **closed simple**, if $\gamma(a) = \gamma(b)$ and $\gamma : [a, b] \to \mathbb{C}$ is otherwise injective;

(4) a **polygonal arc**, if $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ (continuous, concatenated), where each γ_k is a segment parallel to one of the axes.



3 Exact (Path-Independent) Vector Fields

All arcs in the following statements are piecewise continuously differentiable.

Let $p, q : D \to \mathbb{C}$ be continuous on a region D. We call an expression $\omega = p(z)\,dx + q(z)\,dy$ a **differential form**. If this terminology is unfamiliar, simply regard it as notation:

$$\oint_\gamma \omega := \oint_\gamma p(z)\,dx + \oint_\gamma q(z)\,dy.$$

Definition

Let $p, q : D \to \mathbb{C}$ be continuous on a region D. A vector field $\binom{p(z)}{q(z)}$ (equivalently, a differential form $p\,dx + q\,dy$) is **exact**, or **path-independent**, in a region D if for any two arcs $\gamma, \gamma' \subset D$ with the same start point and the same endpoint,

$$\oint_\gamma p(z)\,dx + q(z)\,dy = \oint_{\gamma'} p(z)\,dx + q(z)\,dy.$$

Theorem

The following are equivalent:

(1) $\binom{p}{q}$ is exact.

(2) For every closed $\gamma \subset D$: $\oint_\gamma p\,dx + q\,dy = 0$.

(3) For every closed *polygonal* $\gamma \subset D$: $\oint_\gamma p\,dx + q\,dy = 0$.

(4) There is a real-differentiable $V : D \to \mathbb{C}$ with $\dfrac{\partial V}{\partial x} = p$, $\dfrac{\partial V}{\partial y} = q$ (i.e. $dV := V_x\,dx + V_y\,dy = p\,dx + q\,dy$). V is called a **potential** of $\binom{p}{q}$; $\binom{p}{q}$ is the **gradient field** of V.

Proof. (1) $\Rightarrow$ (2): A closed arc γ from z_0 to z_0 shares endpoints with the trivial (constant) arc at z_0, for which the integral is 0; by path-independence, $\oint_\gamma = 0$.

(2) $\Rightarrow$ (1): If γ, γ' share start and end points, $\gamma - \gamma'$ (i.e. γ followed by γ' reversed) is closed, so by additivity and (2), $0 = \oint_{\gamma-\gamma'} = \oint_\gamma - \oint_{\gamma'}$.

(2) $\Rightarrow$ (3): obvious (restrict to polygonal closed arcs).

(3) $\Rightarrow$ (4): Fix $w_0 \in D$. For $w \in D$, define

$$V(w) := \oint_\gamma p(z)\,dx + q(z)\,dy$$

for *any* polygonal path γ from w_0 to w in D (such a path exists since D is open and connected, hence path-connected by staircase paths). This does not depend on the choice of γ, by (3) (applied to the closed polygonal loop formed by two such choices).

Let $B(w, r) \subset D$ and take $h = c + id$ $(c, d \in \mathbb{R})$, $|h| < r$. Let $\gamma_h = [w, w + c] \cup [w + c, w + h]$, a polygonal path from w to $w + h$. Extending the path defining $V(w)$ by γ_h,

$$V(w+h) - V(w) = \oint_{\gamma_h} p(z)\,dx + q(z)\,dy = \int_0^c p(w+x)\,dx + \int_0^d q(w+c+iy)\,dy.$$

So

$$\begin{aligned}\left|V(w+h) - V(w) - p(w)c - q(w)d\right| \le |c| \sup_{w' \in B(w,2|h|)} |p(w') - p(w)| \\ + |d| \sup_{w' \in B(w,2|h|)} |q(w') - q(w)|.\end{aligned}$$

Here $h \neq 0$ is small enough that $B(w, 2|h|) \subset D$. By continuity of p, q and $|c| + |d| \le \sqrt{2}|h|$, the right-hand side is $o(|h|)$ as $h \to 0$. Thus V is real-differentiable with $\partial V/\partial x = p$ and $\partial V/\partial y = q$.

(4) $\Rightarrow$ (1): For any arc $\gamma : [a, b] \to D$,

$$\oint_\gamma p\,dx + q\,dy = \oint_\gamma \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy = \int_a^b (V \circ \gamma)'(t)\,dt = V(\gamma(b)) - V(\gamma(a)),$$

which depends only on the endpoints, not on γ itself. ■

4 Complex Differentials and Antiderivatives

For $f(z)\,dz = f(z)\,dx + if(z)\,dy$ (i.e. $p = f(z)$, $q = if(z)$): this is exact iff $\exists F$ with $\partial F/\partial x = f(z)$, $\partial F/\partial y = if(z)$, i.e.

$$\frac{\partial F}{\partial z} = \frac{1}{2}(f(z) - i \cdot if(z)) = f(z), \qquad \frac{\partial F}{\partial \bar{z}} = \frac{1}{2}(f(z) + i \cdot if(z)) = 0.$$

So F is complex-differentiable and $F'(z) = f(z)$.

Theorem

$f(z)\,dz$ is exact if and only if there is an analytic F with $F' = f$ (F is called an **antiderivative**). In that case, for any $\gamma \subset D$ from z_1 to z_2,

$$\oint_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Remark (important). Analyticity does not always imply exactness of $f(z)\,dz$. For example, $f(z) = 1/z$ in $D = \mathbb{C} \setminus \{0\}$ satisfies $\oint_\gamma dz/z = 2\pi i \neq 0$ for a counterclockwise circle around 0. Locally, every analytic function has an antiderivative. Below we prove this first under the additional assumption that f' is continuous; the next notes remove that assumption by the Cauchy–Goursat theorem.

5 Rectangle Criterion for Exactness

Throughout this section, rectangles are closed, have sides parallel to the coordinate axes, and their boundaries are oriented counterclockwise. Arcs are piecewise continuously differentiable. Write $\omega = p\,dx + q\,dy$.

Theorem (rectangle criterion in a disc)

Let $\Delta = B(z_0, r)$ and let $p, q : \Delta \to \mathbb{C}$ be continuous. Then ω is exact in Δ if and only if

$$\oint_{\partial R} \omega = 0$$

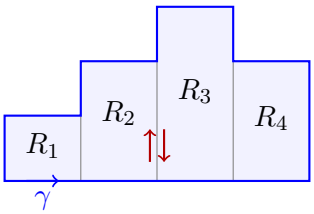
for every rectangle $R \subset \Delta$.

Proof. Necessity follows from exactness. For sufficiency, the earlier equivalence theorem shows that it is enough to prove

$$\oint_\gamma \omega = 0$$

for every closed polygonal arc $\gamma \subset \Delta$ whose segments are parallel to the coordinate axes.

First suppose that γ is simple and oriented counterclockwise. Let P be the polygon together with its interior. The set P lies in the convex hull of γ, and hence in Δ, since a disc is convex. Drawing horizontal and vertical lines through the vertices divides P into finitely many closed rectangles $R_1, \dots, R_N \subset \Delta$ with disjoint interiors.



Each rectangle is oriented counterclockwise; shared edges cancel.

Summing the boundary integrals, each internal edge occurs twice with opposite orientations, leaving only the outer boundary γ. Thus

$$\oint_\gamma \omega = \sum_{j=1}^{N} \oint_{\partial R_j} \omega = 0.$$

For a general closed polygonal arc, subdivide at intersections and at the endpoints of overlapping segments. Splitting the resulting finite closed walk at repeated vertices decomposes it into simple polygonal cycles and pairs of oppositely traversed edges. The cycles have zero integral by the preceding argument, with either orientation, and the opposite edges cancel. Hence every closed polygonal arc has zero integral, so ω is exact in Δ. □

Local consequence. For continuous p, q on an arbitrary open set Ω, the condition $\oint_{\partial R} \omega = 0$ for every rectangle $R \subset \Omega$ implies that ω is exact on every disc contained in Ω. This is *local* exactness; it does not assert exactness on all of Ω.

6 Green's Theorem

Assume $f \in \mathcal{A}(B(z_0, r))$ with f' continuous; Green's Theorem will let us show f has an antiderivative.

Theorem (Green)

Let D be a bounded Jordan domain with piecewise C^1 boundary, oriented counterclockwise. Suppose that p, q are continuous on $\overline{D}$ and that their first partial derivatives exist and are continuous and bounded in D. Then

$$\oint_{\partial D} p\,dx + q\,dy = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy.$$

Sketch for a rectangle $R = [a, b] \times [c, d]$, with $D = \operatorname{Int} R$.

$$\oint_{\partial R} p\,dx + q\,dy = \int_a^b p(x, c)\,dx - \int_a^b p(x, d)\,dx + \int_c^d q(b, y)\,dy - \int_c^d q(a, y)\,dy$$

$$= \int_c^d [q(b, y) - q(a, y)]\,dy - \int_a^b [p(x, d) - p(x, c)]\,dx = \iint_R \frac{\partial q}{\partial x}\,dx\,dy - \iint_R \frac{\partial p}{\partial y}\,dx\,dy,$$

using the Fundamental Theorem of Calculus. First perform the calculation on smaller rectangles inside D, then pass to the limit: continuity on $\overline{D}$ gives convergence of the boundary integrals, and boundedness of the partial derivatives gives convergence of the area integrals. The general case follows by the usual subdivision and boundary-approximation argument.

Complex form. If f is continuous on $\overline{D}$ and has continuous, bounded first partial derivatives in D, then applying Green's Theorem to $p = f$, $q = if$:

$$\oint_{\partial D} f(z)\,dz = \oint_{\partial D} f\,dx + if\,dy = \iint_D \left(\frac{\partial (if)}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = i \iint_D \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy = 2i \iint_D \frac{\partial f}{\partial \bar{z}}\,dx\,dy.$$

Corollary

If $f \in \mathcal{A}(B(z_0, r))$ and f' is continuous, then there is $F \in \mathcal{A}(B(z_0, r))$ with $F' = f$.

Proof. We must show $\oint_{\partial R} f(z)\,dz = 0$ for every rectangle $R \subset B(z_0, r)$. By Green's Theorem (complex form),

$$\oint_{\partial R} f(z)\,dz = 2i \iint_R \frac{\partial f}{\partial \bar{z}}\,dx\,dy = 0,$$

since f is analytic, so $\partial f / \partial \bar{z} \equiv 0$ (Cauchy–Riemann). By the rectangle criterion in a disc, $f(z)\,dz$ is exact on $B(z_0, r)$, giving the antiderivative F. ∎

Classical Cauchy Theorem

If $f \in \mathcal{A}(B(z_0, r))$ and f' is continuous, then for every closed $\gamma \subset B(z_0, r)$,

$$\oint_\gamma f(z)\, dz = 0.$$

(This "classical" version carries the extra hypothesis that f' is continuous; the full Cauchy–Goursat theorem removes it.)